

# Kalman Filtering with Nonlinear State Constraints

Chun Yang  
Sigtem Technology, Inc.  
113 Clover Hill Lane  
Harleysville, PA 19438  
chunyang@sigtem.com

Erik Blasch  
Air Force Research Lab/SNAA  
2241 Avionics Circle  
WPAFB, OH 45433  
erik.blasch@wpafb.af.mil

**Abstract** – In [Simon and Chia, 2002], an analytic method was developed to incorporate linear state equality constraints into the Kalman filter. When the state constraint is nonlinear, linearization was employed to obtain an approximately linear constraint around the current state estimate. This linearized constrained Kalman filter is subject to approximation errors and may suffer from a lack of convergence. In this paper, we present a method that allows exact use of second-order nonlinear state constraints. It is based on a computational algorithm that iteratively finds the Lagrangian multiplier for the nonlinear constraints. The method therefore provides better approximation when higher order nonlinearities are encountered. Computer simulation results are presented to illustrate the algorithm.

**Keywords:** Kalman filtering, nonlinear state constraints, Lagrangian multiplier, iterative solution, tracking.

## 1 Introduction

In a recent paper [Simon and Chia, 2002], a rigorous analytic method was set forth to incorporate linear state equality constraints into the Kalman filtering process. Such constraints (*e.g.*, known model and signal information) are often ignored or dealt with heuristically. The resulting estimates, even obtained with the Kalman filter, cannot be optimal because they do not take advantage of this additional information about state constraints.

One example that benefits from state constraints is the ground target tracking. When a vehicle travels off-road or on an unknown road, the state estimation problem is unconstrained. However, when the vehicle is traveling on a known road, be it straight or curved, the state estimation problem can be cast as constrained with the road network information available from, say, digital terrain maps [Yang, Bakich, and Blasch, 2005].

To make use of state constraints, previous attempts range from reducing the system model parameterization to treating state constraints as perfect measurements. The constrained Kalman filter proposed in [Simon and Chia, 2002] consists of first obtaining an unconstrained Kalman filter solution and then projecting the unconstrained state

estimate onto the constrained surface. Although the main results are restricted to linear systems and linear state equality constraints, the authors outlined steps to extend it to inequality constraints, nonlinear dynamics systems, and nonlinear state constraints.

According to [Simon and Chia, 2002], the inequality constraints can be checked at each time step of the filter. If the inequality constraints are satisfied at a given time step, no action is taken since the inequality constrained problem is solved. If the inequality constraints are not satisfied at a given time step, then the constrained solution is applied to enforce the constraints. Furthermore, to apply the constrained Kalman filter to nonlinear systems and nonlinear state constraints, it is suggested in [Simon and Chia, 2002] to linearize both the system and constraint equations about the current state estimate. The former is equivalent to the use of an extended Kalman filter (EKF).

However, the projection of the unconstrained state estimate onto a linearized state constraint is subject to constraint approximation errors, which is a function of the nonlinearity and more importantly the point around which the linearization takes place. This may result in convergence problems. It was suggested in [Simon and Chia, 2002] to take extra measures to guarantee convergence in the presence of nonlinear constraints.

There are a host of constrained nonlinear optimization techniques [Luenberger, 1989]. Primal methods search through the feasible region determined by the constraints. Penalty and barrier methods approximate constrained optimization problems by unconstrained problems through modifying the objective function (*e.g.*, add a term for higher price if a constraint is violated). Instead of the original constrained problem, dual methods attempt to solve an alternate problem (the dual problem) whose unknowns are the Lagrangian multipliers of the first problem. Cutting plane algorithms work on a series of ever-improving approximating linear programs whose solutions converge to that of the original problem. Lagrangian relaxation methods are widely used in discrete constrained optimization problems.

In this paper, we present a method that allows for the use of second-order nonlinear state constraints exactly. The method can provide better approximation to higher order nonlinearities. The new method is based on a computational algorithm that iteratively finds the

Report Documentation Page			Form Approved OMB No. 0704-0188		
Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.					
1. REPORT DATE <b>JUL 2006</b>		2. REPORT TYPE		3. DATES COVERED <b>00-00-2006 to 00-00-2006</b>	
4. TITLE AND SUBTITLE <b>Kalman Filtering with Nonlinear State Constraints</b>			5a. CONTRACT NUMBER		
			5b. GRANT NUMBER		
			5c. PROGRAM ELEMENT NUMBER		
6. AUTHOR(S)			5d. PROJECT NUMBER		
			5e. TASK NUMBER		
			5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) <b>Air Force Research Lab/SNAA,2241 Avionics Circle,Wright Patterson AFB,OH,45433</b>			8. PERFORMING ORGANIZATION REPORT NUMBER		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)			10. SPONSOR/MONITOR'S ACRONYM(S)		
			11. SPONSOR/MONITOR'S REPORT NUMBER(S)		
12. DISTRIBUTION/AVAILABILITY STATEMENT <b>Approved for public release; distribution unlimited</b>					
13. SUPPLEMENTARY NOTES <b>9th International Conference on Information Fusion, 10-13 July 2006, Florence, Italy. Sponsored by the International Society of Information Fusion (ISIF), Aerospace &amp; Electronic Systems Society (AES), IEEE, ONR, ONR Global, Selex - Sistemi Integrati, Finmeccanica, BAE Systems, TNO, AFOSR's European Office of Aerospace Research and Development, and the NATO Undersea Research Centre.</b>					
14. ABSTRACT <b>see report</b>					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT <b>Same as Report (SAR)</b>	18. NUMBER OF PAGES <b>8</b>	19a. NAME OF RESPONSIBLE PERSON
a. REPORT <b>unclassified</b>	b. ABSTRACT <b>unclassified</b>	c. THIS PAGE <b>unclassified</b>			

Lagrangian multiplier. Considering only second-order constraints in this paper is a tradeoff between reducing approximation errors to higher-order nonlinearities and keeping the problem computationally tractable.

In general, the solution to a filtering problem is an a posteriori probability density function (pdf) and in the linear Gaussian case, it is normally distributed with a mean vector and a covariance matrix. At a first glance, such a solution can never be the solution to a filtering problem with a hard constraint simply because a pdf compliant with a hard constraint cannot have support on the whole state space, as pointed in [Anonym, 2006]. In this paper and in [Simon and Chia, 2002], the constrained projection actually maps the whole state space onto the constraints, producing a constrained pdf with its support on the constraint surface. Although not explicitly in this paper, the constrained pdf can be derived for the second-order constraints in this paper, from which the estimate statistics can in turn be determined albeit approximately.

The paper is organized as follows. Section 2 presents a brief summary of linearly constrained state estimation and its problems encountered when the first-order linearization is used to extend to nonlinear cases. Section 3 details the computational algorithm to solve the second-order constrained least-squared optimization problem. In Section 4, computer simulation results are presented to illustrate the algorithm. Finally, Section 5 provides concluding remarks and suggestions for future work.

## 2 Linear Constrained State Estimation

In this section, we first summarize the results for linearly constrained state estimation [Simon and Chia, 2002] and then point out the problems it may face when the linearization is used to extend it to nonlinear constraints.

Consider a linear time-invariant discrete-time dynamic system together with its measurement as

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k + \underline{w}_k \quad (1a)$$

$$\underline{y}_k = C\underline{x}_k + \underline{v}_k \quad (1b)$$

where the underscore indicates a vector quantity, the subscript  $k$  is the time index,  $\underline{x}$  is the state vector,  $\underline{u}$  is a known input,  $\underline{y}$  is the measurement, and  $\underline{w}$  and  $\underline{v}$  are state and measurement noise processes, respectively. It is implied that all vectors and matrices have compatible dimensions, which are omitted for simplicity.

The goal is to find an estimate denoted by  $\hat{\underline{x}}_k$  of  $\underline{x}_k$  given the measurements up to time  $k$  denoted by  $Y_k = \{\underline{y}_0, \dots, \underline{y}_k\}$ . Under the assumptions that the state and measurement noises are uncorrelated zero-mean white Gaussian with  $\underline{w} \sim \mathcal{M}(0, Q)$  and  $\underline{v} \sim \mathcal{M}(0, R)$  where  $Q$  and  $R$  are positive semi-definite covariance matrices, the Kalman filter provides an optimal estimator in the form of  $\hat{\underline{x}}_k = E\{\underline{x}_k | Y_k\}$  [Anderson and Moore, 1979]. Starting

from an initial estimate  $\bar{\underline{x}}_0 = E\{\underline{x}_0\}$  and its estimation error covariance matrix  $P_0 = E\{(\underline{x}_0 - \bar{\underline{x}}_0)(\underline{x}_0 - \bar{\underline{x}}_0)^T\}$  where the superscript  $T$  stands for matrix transpose, the Kalman filter equations specify the propagation of  $\hat{\underline{x}}_k$  and  $P_k$  over time and the update of  $\hat{\underline{x}}_k$  and  $P_k$  by measurement  $\underline{y}_k$  as

$$\hat{\underline{x}}_{k+1} = A\hat{\underline{x}}_k + B\underline{u}_k + K_k(\underline{y}_k - C\hat{\underline{x}}_k) \quad (2a)$$

$$K_k = AP_k C^T (CP_k C^T + R)^{-1} \quad (2b)$$

$$P_{k+1} = (AP_k - K_k CP_k)A^T + Q \quad (2c)$$

Now in addition to the dynamic system of Eq. (1), we are given the linear state constraint

$$D\underline{x}_k = \underline{d} \quad (3)$$

where  $D$  is a known constant matrix of full rank,  $\underline{d}$  is a known vector, and the number of rows in  $D$  is the number of constraints, which is assumed to be less than the number of states. If  $D$  is a square matrix, the state is fully constrained and can thus be solved by inverting Eq. (3). Although no time index is given to  $D$  and  $\underline{d}$  in Eq. (3), it is implied that they can be time-dependent, leading to piecewise linear constraints.

The constrained Kalman filter according to [Simon and Chia, 2002] is constructed by directly projecting the unconstrained state estimate  $\hat{\underline{x}}_k$  onto the constrained surface  $S = \{\underline{x}: D\underline{x} = \underline{d}\}$ . It is formulated as the solution to the problem:

$$\tilde{\underline{x}} = \arg \min_{\underline{x} \in S} (\underline{x} - \hat{\underline{x}})^T W (\underline{x} - \hat{\underline{x}}) \quad (4)$$

where  $W$  is a symmetric positive definite weighting matrix. To solve this problem, we form the Lagrangian

$$J(\underline{x}, \underline{\lambda}) = (\underline{x} - \hat{\underline{x}})^T W (\underline{x} - \hat{\underline{x}}) + 2\underline{\lambda}^T (D\underline{x} - \underline{d}) \quad (5)$$

The first order conditions necessary for a minimum are given by

$$\frac{\partial J}{\partial \underline{x}} = 0 \Rightarrow W(\underline{x} - \hat{\underline{x}}) + D^T \underline{\lambda} = 0 \quad (6a)$$

$$\frac{\partial J}{\partial \underline{\lambda}} = 0 \Rightarrow D\underline{x} - \underline{d} = 0 \quad (6b)$$

This gives the solution

$$\underline{\lambda} = (DW^{-1}D^T)^{-1}(D\hat{\underline{x}} - \underline{d}) \quad (7a)$$

$$\tilde{\underline{x}} = \hat{\underline{x}} - W^{-1}D^T(DW^{-1}D^T)^{-1}(D\hat{\underline{x}} - \underline{d}) \quad (7b)$$

The above derivation does not depend on the conditional Gaussian nature of the unconstrained estimate  $\hat{\underline{x}}$ . It was shown in [Simon and Chia, 2002] that when  $W = I$ , the solution in Eq. (7) is the same as what is obtained by the mean square method, which attempts to minimize the

conditional mean square error subject to the state constraints, that is,

$$\min_{\underline{\hat{x}}} E\{\|\underline{x} - \underline{\hat{x}}\|_2^2 | Y\} \quad \text{such that } D\underline{\hat{x}} = \underline{d} \quad (8)$$

where  $\|\cdot\|_2$  denotes the vector two-norm. Furthermore, when  $W = P^{-1}$ , i.e., the inverse of the unconstrained state estimation error covariance, the solution in Eq. (7) reduces to the result given by the maximum conditional probability method

$$\max_{\underline{\hat{x}}} \ln \text{Prob}\{\underline{x} | Y\} \quad \text{such that } D\underline{\hat{x}} = \underline{d} \quad (9)$$

Several interesting statistical properties of the constrained Kalman filter are presented in [Simon and Chia, 2002]. This includes the fact that the constrained state estimate as given by Eq. (7) is an unbiased state estimate for the system in Eq. (1) subject to the constraint in Eq. (3) for a known symmetric positive definite weighting matrix  $W$ . Furthermore when  $W = P^{-1}$ , the constrained state estimate has a smaller error covariance than that of the unconstrained state estimate, and it is actually the smallest for all constrained Kalman filters of this type. Similar results hold in terms of the trace of the estimation error covariance matrix when  $W = I$ .

In practical applications, however, nonlinear state constraints are likely to emerge. Consider the nonlinear state constraint of the form

$$g(\underline{x}) = \underline{d} \quad (10)$$

We can expand the nonlinear state constraints about a constrained state estimate  $\underline{\tilde{x}}$  and for the  $i^{\text{th}}$  row of Eq. (10), we have

$$g_i(\underline{x}) - d_i = g_i(\underline{\tilde{x}}) + g_i'(\underline{\tilde{x}})^T (\underline{x} - \underline{\tilde{x}}) + \frac{1}{2!} (\underline{x} - \underline{\tilde{x}})^T g_i''(\underline{\tilde{x}}) (\underline{x} - \underline{\tilde{x}}) + \dots - d_i = 0 \quad (11)$$

where the superscripts ' and '' denote the first and second partial derivatives.

Keeping only the first-order terms as suggested in [Simon and Chia, 2002], some rearrangement leads to

$$g'(\underline{\tilde{x}})^T \underline{x} \approx \underline{d} - g(\underline{\tilde{x}}) + g'(\underline{\tilde{x}})^T \underline{\tilde{x}} \quad (12)$$

where  $g(\underline{x}) = [\dots g_i(\underline{x}) \dots]^T$ ,  $\underline{d} = [\dots d_i \dots]^T$ , and  $g'(\underline{x}) = [\dots g_i'(\underline{x}) \dots]^T$ . An approximate linear constraint is therefore formed by replacing  $D$  and  $\underline{d}$  in Eq. (3) with  $g'(\underline{x})$  and  $\underline{d} - g(\underline{\tilde{x}}) + g'(\underline{\tilde{x}})^T \underline{\tilde{x}}$ , respectively.

Figure 1 illustrates this linearization process, which identifies possible errors associated with linear approximation of a nonlinear state constraint. As shown, the previous constrained state estimate  $\underline{\tilde{x}}^-$  lies somewhere on the constrained surface but is away from the true state. The projection of the unconstrained state estimate  $\underline{\hat{x}}$  onto

the approximate linear state constraint produces the current constrained state estimate  $\underline{\tilde{x}}^+$ , which is however subject to the constraint approximation error. Clearly, the further away is  $\underline{\tilde{x}}^-$  from  $\underline{x}$ , the larger is the approximation-introduced error. More critically, such an approximately linear constrained estimate may not satisfy the original nonlinear constraint specified in Eq. (10). It is therefore desired to reduce this approximation-introduced error by including higher-order terms while keeping the problem computationally tractable. One possible approach is presented in the next section.

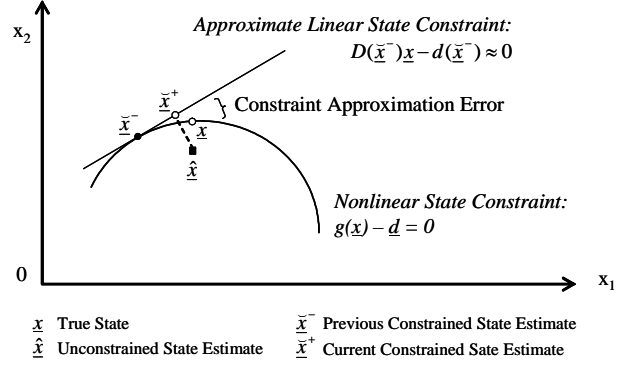


Figure 1 – Errors in Linear Approximation of Nonlinear State Constraints

### 3 Nonlinear Constrained Estimation

In this section, we consider a second-order state constraint function, which can be viewed as a second-order approximation to an arbitrary nonlinearity, as

$$f(\underline{x}) = \begin{bmatrix} \underline{x}^T & 1 \end{bmatrix} \begin{bmatrix} M & \underline{m} \\ \underline{m}^T & m_0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix} = \underline{x}^T M \underline{x} + \underline{m}^T \underline{x} + \underline{x}^T \underline{m} + m_0 = 0 \quad (13)$$

As an example, a general equation of the second degree in two variables  $\xi$  and  $\eta$  is written as

$$f(\xi, \eta) = a\xi^2 + 2b\xi\eta + c\eta^2 + 2d\xi + 2e\eta + f = \begin{bmatrix} \xi & \eta & 1 \end{bmatrix} \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ 1 \end{bmatrix} = 0 \quad (14)$$

which may represent a road segment in a digital terrain map.

Following the constrained Kalman filtering of [Simon and Chia, 2002], we can formulate the projection of an unconstrained state estimation onto a nonlinear constraint surface as the constrained least-square optimization problem

$$\underline{\hat{x}} = \arg \min_{\underline{x}} (\underline{z} - H\underline{x})^T (\underline{z} - H\underline{x}) \quad (15)$$

subject to  $f(\underline{x}) = 0$

If we let  $W = H^T H$  and  $z = H \hat{x}$ , the formulation in Eq. (15) becomes the same as in Eq. (4). In a sense, Eq. (15) is a more general formulation because it can also be interpreted as a nonlinear constrained measurement update or a projection in the predicted measurement domain.

Construct the Lagrangian with the Lagrangian multiplier  $\lambda$  as

$$J(\underline{x}, \lambda) = (\underline{z} - H\underline{x})^T (\underline{z} - H\underline{x}) + \lambda f(\underline{x}) \quad (17)$$

Taking the partial derivatives of  $J(\underline{x}, \lambda)$  with respect to  $\underline{x}$  and  $\lambda$ , respectively, setting them to zero leads to the necessary conditions:

$$-H^T \underline{z} + \lambda \underline{m} + (H^T H + \lambda M) \underline{x} = 0 \quad (18a)$$

$$\underline{x}^T M \underline{x} + \underline{m}^T \underline{x} + \underline{x}^T \underline{m} + m_0 = 0 \quad (18b)$$

Assume that the inverse matrix of  $H^T H + \lambda M$  exists. Then,  $\underline{x}$  can be solved from Eq. (18a), giving the constrained solution in terms of the unknown  $\lambda$  as

$$\underline{x} = (H^T H + \lambda M)^{-1} (H^T \underline{z} - \lambda \underline{m}) \quad (19)$$

which reduces to the unconstrained least-squares solution when  $\lambda = 0$ .

Assume that the matrix  $M$  admits the factorization  $M = L^T L$  and apply the Cholesky factorization to  $W = H^T H$  as

$$W (= H^T H) = G^T G \quad (20)$$

where  $G$  is an upper right diagonal matrix. We then perform a singular value decomposition (SVD) of the matrix  $LG^{-1}$  [Moon and Stirling, 2000] as

$$LG^{-1} = U \Sigma V^T \quad (21)$$

where  $U$  and  $V$  are orthonormal matrices and  $\Sigma$  is a diagonal matrix with its diagonal elements denoted by  $\sigma_i$ . For a square matrix, this becomes the eigenvalue decomposition.

Introduce two new vectors

$$\underline{e}(\lambda) = [\dots e_i(\lambda), \dots]^T = V^T (G^T)^{-1} (H^T \underline{z} - \lambda \underline{m}) \quad (22a)$$

$$\underline{t} = [\dots t_i \dots]^T = V^T (G^T)^{-1} \underline{m} \quad (22b)$$

With these factorizations and new matrix and vector notations, the constrained solution in Eq. (19) can be simplified as

$$\underline{x} = G^{-1} V (I + \lambda \Sigma^T \Sigma)^{-1} \underline{e}(\lambda) \quad (23)$$

The first and second order terms in  $\underline{x}$  in Eq. (18b) can be expressed in  $\lambda$  as:

$$\underline{x}^T M \underline{x} = \underline{e}(\lambda)^T (I + \lambda \Sigma^T \Sigma)^{-T} \Sigma^T \Sigma (I + \lambda \Sigma^T \Sigma)^{-1} \underline{e}(\lambda)$$

$$= \sum_i \frac{e_i^2(\lambda) \sigma_i^2}{(1 + \lambda \sigma_i^2)^2} \quad (23a)$$

$$\underline{m}^T \underline{x} = \underline{t}^T (I + \lambda \Sigma^T \Sigma)^{-1} \underline{e}(\lambda) = \sum_i \frac{e_i(\lambda) t_i}{1 + \lambda \sigma_i^2} \quad (23b)$$

$$\underline{x}^T \underline{m} = \underline{e}(\lambda)^T (I + \lambda \Sigma^T \Sigma)^{-1} \underline{t} = \sum_i \frac{e_i(\lambda) t_i}{1 + \lambda \sigma_i^2} \quad (23c)$$

Bringing these terms into the constrained equation in Eq. (18b) gives rise to the constraint equation, now expressed in terms of the unknown Lagrangian multiplier  $\lambda$ , as

$$\begin{aligned} f(\lambda) &= (\underline{z}^T H - \lambda \underline{m}^T) (H^T H + \lambda M)^{-1} (H^T \underline{z} - \lambda \underline{m}) \\ &\quad + \underline{m}^T (H^T H + \lambda M)^{-1} (H^T \underline{z} - \lambda \underline{m}) \\ &\quad + (\underline{z}^T H - \lambda \underline{m}^T) (H^T H + \lambda M)^{-1} \underline{m} + m_0 \\ &= \underline{e}(\lambda)^T (I + \lambda \Sigma^T \Sigma)^{-1} \Sigma^T \Sigma (I + \lambda \Sigma^T \Sigma)^{-1} \underline{e}(\lambda) \\ &\quad + \underline{t}^T (I + \lambda \Sigma^T \Sigma)^{-1} \underline{e}(\lambda) + \underline{e}(\lambda)^T (I + \lambda \Sigma^T \Sigma)^{-1} \underline{t} + m_0 \\ &= \sum_i \frac{e_i^2(\lambda) \sigma_i^2}{(1 + \lambda \sigma_i^2)^2} + 2 \sum_i \frac{e_i(\lambda) t_i}{1 + \lambda \sigma_i^2} + m_0 \end{aligned} \quad (24)$$

As a nonlinear equation in  $\lambda$ , it is difficult to find a closed-form solution in general. Numerical root-finding algorithms may be used instead. For example, the Newton's method is used below. Denote the derivative of  $f(\lambda)$  with respect to  $\lambda$  as

$$\begin{aligned} \dot{f}(\lambda) &= 2 \sum_i \frac{e_i(\lambda) \dot{e}_i (1 + \lambda \sigma_i^2) \sigma_i^2 - e_i^2(\lambda) \sigma_i^4}{(1 + \lambda \sigma_i^2)^3} \\ &\quad + 2 \sum_i \frac{\dot{e}_i t_i (1 + \lambda \sigma_i^2) - e_i(\lambda) t_i \sigma_i^2}{(1 + \lambda \sigma_i^2)^2} \end{aligned} \quad (25a)$$

where

$$\dot{\underline{e}} = [\dots \dot{e}_i \dots]^T = -V^T (G^T)^{-1} \underline{m} \quad (25b)$$

Then the iterative solution for  $\lambda$  is given by

$$\lambda_{k+1} = \lambda_k - \frac{f(\lambda_k)}{\dot{f}(\lambda_k)}, \quad \text{starting with } \lambda_0 = 0 \quad (26)$$

The iteration stops when  $|\lambda_{k+1} - \lambda_k| < \tau$ , a given small value or the number of iterations reaches a pre-specified number. Then bringing the converged Lagrangian multiplier  $\lambda$  back to Eq. (19) or (23) provides the constrained optimal solution.

Now consider the special case where  $W = H^T H$ ,  $z = H \hat{x}$ , and  $\underline{m} = 0$ , that is, a quadratic constraint on the state. Under these conditions,  $\underline{t} = 0$  and  $\underline{e}$  is no longer a function of  $\lambda$  so its derivative relative to  $\lambda$  vanishes,  $\dot{\underline{e}} = 0$ . The quadratic constrained solution is then given by

$$\tilde{\underline{x}} = (W + \lambda M)^{-1} W \hat{x} \quad (27a)$$

where the Lagrangian multiplier  $\lambda$  is obtained iteratively as in Eq. (26) with the corresponding  $f(\lambda)$  and  $\dot{f}(\lambda)$  given by

$$f(\lambda) = \sum_i \frac{e_i^2 \sigma_i^2}{(1 + \lambda \sigma_i^2)^2} + m_0 \quad (27b)$$

$$\dot{f}(\lambda) = -2 \sum_i \frac{e_i^2 \sigma_i^4}{(1 + \lambda \sigma_i^2)^3} \quad (27c)$$

The solution of Eq. (27) is also called the constrained least squares [Moon and Stirling, 2000; pp 765-766], which was previously applied for the joint estimation and calibration [Yang and Lin, 2004]. When  $M = 0$ , the constraint in Eq. (13) degenerates to a linear one. The constrained solution in Eq. (19) is still valid. However, the iterative solution for finding  $\lambda$  is no longer applicable and a closed-form solution is available as given in Eq. (7).

## 4 Simulation Results

In this section, a simple example is used to demonstrate the effectiveness of the nonlinear constrained method of this paper and to show its superior performance as compared to the linearized constrained method in [Simon and Chia, 2002]. In this example, a ground vehicle is assumed to travel along a circular road segment as shown in Figure 2 with the turn center chosen as the origin of the x-y coordinates.

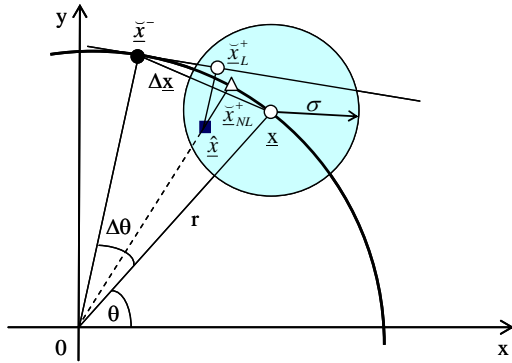


Figure 2 – Tracking a Vehicle along a Circular Road

Under this assumption, the road constraint is quadratic and can be written as

$$f(x, y) = x^2 + y^2 - r^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - r^2 = 0 \quad (28)$$

Assume that the true position of the vehicle is at  $\underline{x} = [r \cos \theta, r \sin \theta]^T$  and an unbiased estimate denoted by  $\hat{\underline{x}}$  is obtained by a certain means such that

$$\hat{\underline{x}} \sim N(\underline{x}, P) \quad (29)$$

where  $P$  is the estimation error covariance matrix, which is assumed to take a diagonal form as  $P = \text{diag}\{\sigma_x^2, \sigma_y^2\}$  with  $\sigma_x^2 = \sigma_y^2 = \sigma^2$ . It is understandable that additional errors will appear if the estimate  $\hat{\underline{x}}$  is biased.

Furthermore, if the error covariance matrix is not diagonal, the correlation direction will also affect the statistical property. Ruling out such variability in conditions will make results analysis easier while not losing the generality.

To apply the linear constrained Kalman filter of [Simon and Chia, 2002], the nonlinear constraint is linearized about a previous constrained state denoted by  $\tilde{\underline{x}}^-$  and can be written as

$$\begin{bmatrix} \tilde{\underline{x}}^- & \tilde{\underline{y}}^- \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = r^2 \quad (30)$$

In terms of the matrix notations, we have  $D = \begin{bmatrix} \tilde{\underline{x}}^- & \tilde{\underline{y}}^- \end{bmatrix}$  and  $d = r^2$ . Since the previous constrained state  $\tilde{\underline{x}}^-$  is also on the circular path, it can be parameterized as  $\tilde{\underline{x}}^- = [r \cos(\theta + \Delta\theta), r \sin(\theta + \Delta\theta)]^T$  where  $\Delta\theta$  is the angular offset and its distance to the true state is given by

$$\|\Delta \underline{x}\|_2^2 = \|\tilde{\underline{x}}^- - \underline{x}\|_2^2 = 2r^2(1 - \cos \Delta\theta) \quad (31)$$

With  $W = P^{-1}$  (i.e.,  $H = \sigma^{-1}I_2$  and  $\underline{z} = H\hat{\underline{x}}$ , where  $I_n$  stands for an identity matrix of dimension  $n$ ), the linearized constrained estimate denoted by  $\tilde{\underline{x}}_L^+$  can be calculated using Eq. (7).

Similarly, with  $W = P^{-1}$ ,  $M = I_2$ , and  $m_0 = r^2$ , the quadratic constrained estimate denoted by  $\tilde{\underline{x}}_{NL}^+$  is given by Eq. (27)

In the first simulation, we set  $r = 100$  m,  $\theta = 45^\circ$ , and  $\sigma = 10$  m and then draw samples of  $\hat{\underline{x}}$  from the distribution in Eq. (29) as the unconstrained state estimates. We then calculate both the linearized and nonlinear constrained state estimates  $\tilde{\underline{x}}_L^+$  and  $\tilde{\underline{x}}_{NL}^+$  at two linearizing points, i.e., two previous constrained state estimates  $\tilde{\underline{x}}^-$ , with  $\Delta\theta = 0^\circ$  and  $-15^\circ$ , respectively.

Figures 3 and 4 illustrate how the nonlinear constrained method of this paper and the linearized constrained method of [Simon and Chia, 2002] actually project unconstrained state estimates onto a nonlinear constraint, which is a circular road segment in this example. In Figure 3 for  $\Delta\theta = 0^\circ$ , the linearizing point (small circle o) coincides with the true state (star \*) (the estimator is not aware of, though). The linearized constraint is a line tangent to the circular path at the point. There are 50 random samples drawn from the distribution of Eq. (29) as the unconstrained state estimates (dot ·), which are projected onto the linearized constraint using Eq. (7) as the linearized constrained estimates (cross sign x) and using Eq. (27) as the quadratic constrained estimates (plus sign +). Clearly, all the quadratic constrained estimates fall onto the circle, thus satisfying the constraint whereas not all linearized constrained estimates do so.

In Figure 4 for  $\Delta\theta = -15^\circ$ , the linearizing point is away from the true state. At 100 m, an angular offset is related

to a separation error by a factor of 1.74 m/deg. This corresponds to about 26 m for  $\Delta\theta = -15^\circ$ . Although the linearized method using Eq. (7) effectively projects all the unconstrained state estimates onto the linearized constraint, the linearized constraint itself is tilted off, introducing larger errors due to the constraint approximation. In contrast, the quadratic constrained method, being independent of the prior knowledge about the state estimate, easily satisfies the constraint.

In the second simulation, we draw 5000 unconstrained state estimates for each linearizing point  $\Delta\theta$ , which varies from  $-20^\circ$  to  $20^\circ$  in steps of  $5^\circ$ . At each linearizing point, we calculate the root mean squared (RMS) errors between the true state and the linearized and nonlinear constrained state estimates, respectively. Figure 5 shows the two RMS values as a function of the linearizing point together with the unconstrained state estimation error standard deviation  $\sigma$ .

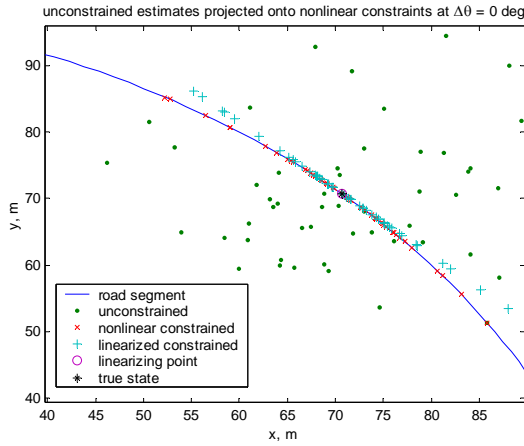


Figure 3 – Projection of Unconstrained Estimates onto Constraints ( $\tilde{\mathbf{x}}^- = \mathbf{x}$ )

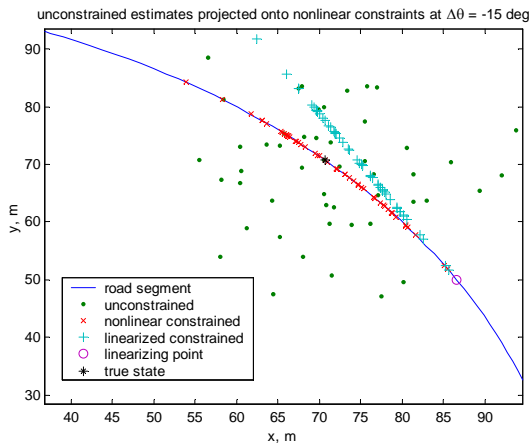


Figure 4 – Projection of Unconstrained Estimates onto Constraints ( $\tilde{\mathbf{x}}^- \neq \mathbf{x}$ )

As expected, the RMS values for the linearized constrained estimates (circle o) grow large as the angular offset  $\Delta\theta$  increases. The RMS values for the quadratic constrained estimates (dot ·) remain constant and are

slightly smaller than the unconstrained estimates, because the scattering along a constraint curve is smaller than in a two dimensional (2D) plan. For the same reason, the RMS values for the linearized constrained estimates are smaller than those of the quadratic constrained estimates for  $|\Delta\theta| < 7^\circ$ . However, the use of RMS values for comparison is less meaningful in this case. As shown in Figure 6, both the unconstrained estimates (circle o) and linearized constrained estimates (dot ·) do not always satisfy the nonlinear constraints whereas quadratic constrained estimates (plus sign +) do all the time (i.e., the constraint is satisfied if the distance to the origin is 100 m).

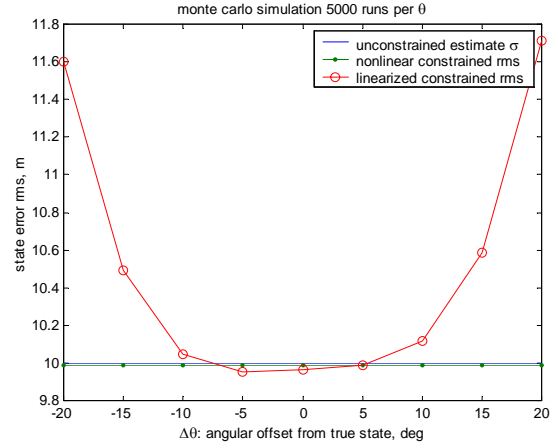


Figure 5 – RMS Values of State Estimates vs. Angular Offset

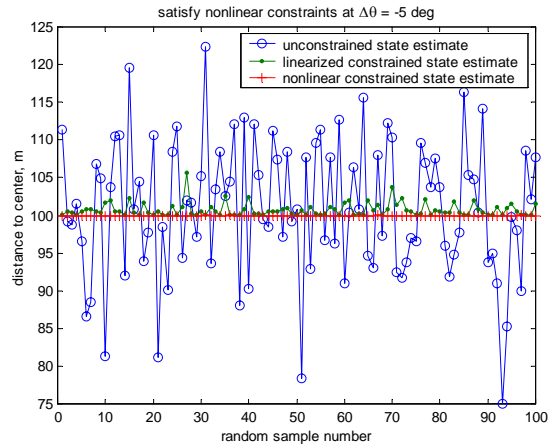


Figure 6 – Constraint Satisfaction

In a 2D setting, the circular path constraint of Eq. (30) allows for a simple geometry solution to the problem. The desired on-circle point  $\tilde{\mathbf{x}}$  is the intersection between the circle and the line extending from the origin to the off-circle point  $\hat{\mathbf{x}}$ . The geometry solution can be written as

$$\tilde{x} = r \cos[\tan^{-1}(\frac{\hat{y}}{\hat{x}})] \quad (32a)$$

$$\tilde{y} = r \sin[\tan^{-1}(\frac{\hat{y}}{\hat{x}})] \quad (32b)$$

Since it is an exact solution for this particular simulation example, we can use it to verify the iterative solution obtained by the quadratic constrained method. Figure 7 shows the quadratic constrained estimates (cross x) obtained by the iterative algorithm of Eq. (27) and the exact geometry solutions (circle o), which indeed match each other perfectly.

Finally, we use Figure 8 to show an example of the Lagrangian multiplier as it is calculated iteratively using Eq. (27). The runs for five unconstrained state estimates are plotted in the same figure and to make it fit, the normalized absolute values of  $\lambda$  are taken. As shown, starting from zero, it typically takes 4 iterations for the algorithm to converge in the example presented.

In fact, for this simple example of a target traveling along a circle used in the simulation, a closed-form solution can be derived. Assume that  $W = I_2$ ,  $M = I_2$ ,  $\underline{m} = 0$ , and  $m_0 = -r^2$ . The nonlinear constraint can be equivalently written as:

$$\underline{x}^T \underline{x} = r^2 \quad (33)$$

The quadratic constrained estimate given in Eq. (27a) is repeated below for easy reference:

$$\tilde{\underline{x}} = (W + \lambda M)^{-1} W \hat{\underline{x}} = (1 + \lambda)^{-1} \hat{\underline{x}} \quad (34)$$

where  $\lambda$  is the Lagrangian multiplier.

Bringing Eq. (34) back to Eq. (33) gives:

$$\tilde{\underline{x}}^T \tilde{\underline{x}} = \left( \frac{\hat{\underline{x}}}{1 + \lambda} \right)^T \frac{\hat{\underline{x}}}{1 + \lambda} = r^2 \quad (35)$$

The solution for  $\lambda$  is:

$$\lambda = \frac{\sqrt{\hat{\underline{x}}^T \hat{\underline{x}}} - 1}{r} = \frac{\|\hat{\underline{x}}\|_2}{r} - 1 \quad (36)$$

where  $\|\cdot\|_2$  stands for the 2-norm or length for the vector.

Bringing the solution for  $\lambda$  in Eq. (36) back to Eq. (34) gives:

$$\tilde{\underline{x}} = r \frac{\hat{\underline{x}}}{\|\hat{\underline{x}}\|_2}$$

This indicates that for this particular case, the constraining results in normalization. This is consistent with the geometric solution shown in the above simulation.

This suggests a simple solution for some practical applications. When a target is traveling along a circular path (or approximately so), one can first find the equivalent center of the circle around which to establish a new coordinate system. Then express the unconstrained solution in the new coordinate and normalize it as the constrained solution. Finally convert it back to the original coordinates. For non-circular but second-order paths, eigenvalue-based scaling may be effected following

coordinate translation and rotation in order to apply this circular normalization. Reverse operations are then used to transform back. For applications of high dimensionality, the scalar iterative solution of Eq. (26) may be more efficient.

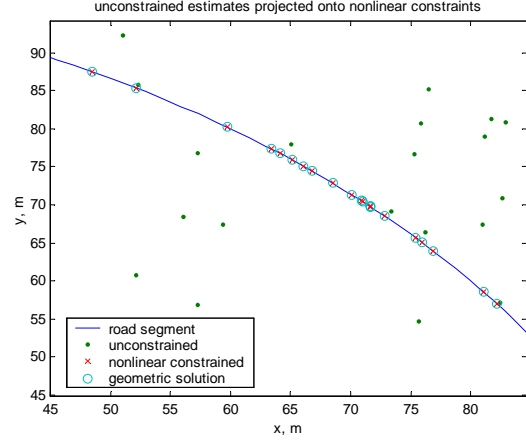


Figure 7 – Iterative Quadratic Constrained Solution vs. Geometry Solution

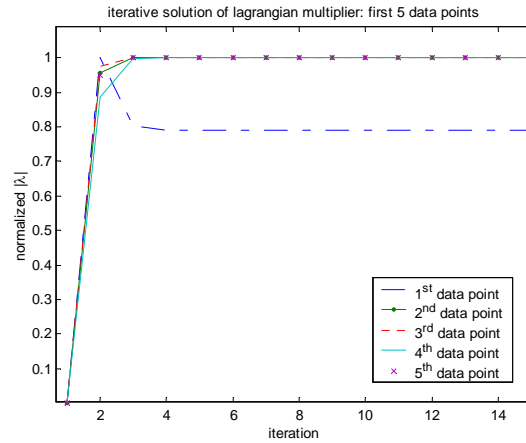


Figure 8 – Convergence in Iterative Lagrangian Multiplier

## 5 Conclusions

In this paper, we have presented a method for incorporating second-order nonlinear state constraints in a Kalman filter. It is generalized from our earlier solutions for quadratic constraint functions to more general second-order state constraint functions. It can be viewed as an extension of the Kalman filter with linear state equality constraints to nonlinear cases. Simulation results demonstrate the performance of this method as compared to linearized state constraints.

Although the present solution considers a scalar constraint, it is possible to extend the solution to cases where multiple nonlinear constraints or hybrid linear and nonlinear constraints are imposed. Other directions for future work include search for more efficient root finding algorithms to solve for the Lagrangian multiplier. With a



quadratic cost function, it is of interest to further extend the iterative method to explore other types of nonlinear constraints of practical significance.

## Acknowledgements

Research supported in part under Contracts No. FA8650-04-M-1624 and FA8650-05-C-1808, which is gratefully acknowledged. Thanks also go to Dr. Rob Williams, Dr. Devert Wicker, and Dr. Jeff Layne for their valuable discussions and to Dr. Mike Bakich for his assistance.

## References

- Anonym, Comments from Reviewer # 1 of Fusion2006 Technical Review Committee, April 2006.
- B. Anderson and J. Moore, *Optimal Filtering*, Prentice Hall, Englewood Cliffs, NJ, 1979.
- D.G. Luenberger, *Linear and Nonlinear Programming* (2<sup>nd</sup> Ed.), Addison-Wesley, 1989.
- T.K. Moon and W.C. Stirling, *Mathematical Methods and Algorithms for Signal Processing*, Prentice Hall, Upper Saddle River, NJ, 2000.
- D. Simon and T.L. Chia, Kalman Filtering with State Equality Constraints, *IEEE Trans. On Aerospace and Electronic Systems*, 38, 1 (Jan. 2002), 128-136.
- C. Yang, M. Bakich, and E. Blasch, Nonlinear Constrained Tracking of Targets on Roads, *Proc. of the 8<sup>th</sup> International Conf. on Information Fusion*, Philadelphia, PA, July 2005.
- C. Yang and D.M. Lin, Constrained Optimization for Joint Estimation of Channel Biases and Angles of Arrival for Small GPS Antenna Arrays, *Proc. of the 60<sup>th</sup> Annual Meeting of the Institute of Navigation*, Dayton, OH, 2004.